On calculation of the interweight distribution of an equitable partition

Denis Krotov

Sobolev Institute of Mathematics, Novosibirsk, Russia

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Outline

- Definitions
- Examples of equitable partitions
- Distance invariance of e. p.
- Interweight distribution and strong distance invariance
- Calculating weight distribution of e. p.
- Calculating interweight distribution of e. p.
- Problems and conclusion
- Thank you

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Equitable partitions

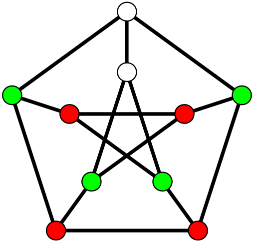
Let $G = (V(G), E(G))$ be a graph.

Definition

A partition $(C_1, \ldots, C_k)$ of $V(G)$ is an equitable partition with quotient matrix $S = (S_{ij})_{i,j=1}^{k}$ iff every element of C_i is adjacent with exactly S_{ij} elements of C_j.

Equitable partitions $\sim$ regular partitions $\sim$ partition designs $\sim$ perfect colorings $\sim \ldots$

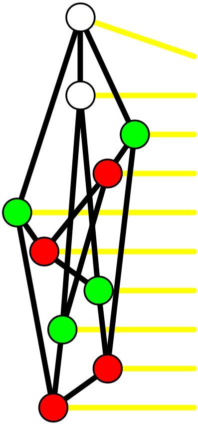
Example: Equitable partition



$$S = \begin{array}{c|ccc} & \circ & {\color{green}\bullet} & {\color{red}\bullet} \\ \hline \circ & 1 & 2 & 0 \\ {\color{green}\bullet} & 1 & 0 & 2 \\ {\color{red}\bullet} & 0 & 2 & 1 \end{array}$$

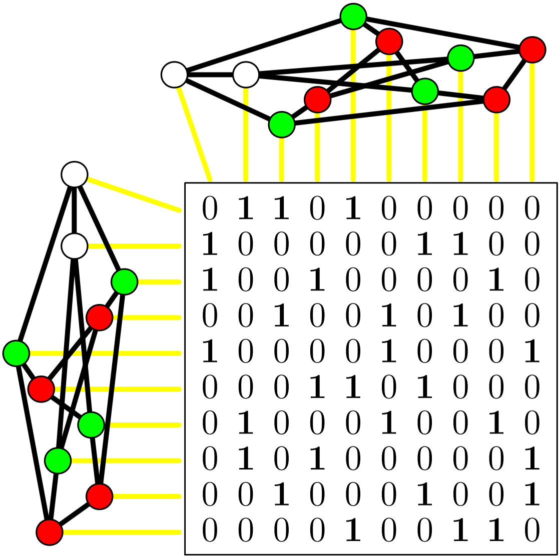
Incidence matrix of an equitable partition

$\overline{C}$:



$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Adjacency matrix



$$A: \quad \begin{bmatrix} 0&1&1&0&1&0&0&0&0&0\\ 1&0&0&0&0&0&1&1&0&0\\ 1&0&0&1&0&0&0&0&1&0\\ 0&0&1&0&0&1&0&1&0&0\\ 1&0&0&0&0&1&0&0&0&1\\ 0&0&0&1&1&0&1&0&0&0\\ 0&1&0&0&0&1&0&0&1&0\\ 0&1&0&1&0&0&0&0&0&1\\ 0&0&1&0&0&0&1&0&0&1\\ 0&0&0&0&1&0&0&1&1&0 \end{bmatrix}$$

Matrix equation for equitable partition

A – adjacency matrix of the graph; $\overline{C}$ – incidence matrix of an equitable partition with quotient matrix S.

$$A\overline{C} = \overline{C}S$$

$$\begin{pmatrix} 0&1&1&0&1&0&0&0&0&0\\ 1&0&0&0&0&0&1&1&0&0\\ 1&0&0&1&0&0&0&0&1&0\\ 0&0&1&0&0&1&0&1&0&0\\ 1&0&0&0&0&1&0&0&0&1\\ 0&0&0&1&1&0&1&0&0&0\\ 0&1&0&0&0&1&0&0&1&0\\ 0&1&0&1&0&0&0&0&0&1\\ 0&0&1&0&0&0&1&0&0&1\\ 0&0&0&0&1&0&0&1&1&0 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0\\ 1&0&0\\ 0&1&0\\ 0&0&1\\ 0&1&0\\ 0&0&1\\ 0&1&0\\ 0&1&0\\ 0&0&1\\ 0&0&1 \end{pmatrix} = \begin{pmatrix} 1&0&0\\ 1&0&0\\ 0&1&0\\ 0&0&1\\ 0&1&0\\ 0&0&1\\ 0&1&0\\ 0&1&0\\ 0&0&1\\ 0&0&1 \end{pmatrix} \cdot \begin{pmatrix} 1&2&0\\ 1&0&2\\ 0&2&1 \end{pmatrix}$$

Equitable partitions in coding theory

- Coset partition of a linear (Z4-linear) code is an equitable partition.
- Distance partition of a completely regular code is an equitable partition (with 3-diagonal quotient matrix). E.g., perfect codes, nearly perfect codes.
- 1-perfect codes: $\begin{pmatrix} 0 & d \\ 1 & d-1 \end{pmatrix}$
- distance-5 Preparata-like codes: $\begin{pmatrix} 0 & n & 0 & 0 \\ 1 & 0 & n-1 & 0 \\ 0 & 2 & n-3 & 1 \\ 0 & 0 & n & 0 \end{pmatrix}$
- Any distance-3 code of cardinality $2^n/(n+3)$:
$$\begin{pmatrix} 0 & 1 & n-1 & 0 \\ 1 & 0 & n-1 & 0 \\ 1 & 1 & n-4 & 2 \\ 0 & 0 & n-1 & 1 \end{pmatrix}$$

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Distance invariance of equitable partitions

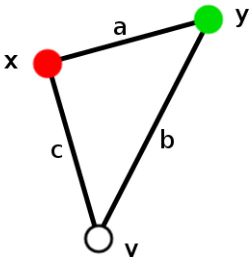
Equitable partitions of distance-regular graph are distance invariant:

Distance invariance (equitable partition)

The weight distributions of the cells $C_1, \ldots, C_k$ with respect to a vertex $\overline{x} \in C_i$ depend only on i (do not depend on the choice of $\overline{x}$ in C_i).

Interweight distribution

- One of the most strong known generalizations of the distance invariance of equitable partitions is the strong distance invariance.
- For a fixed vertex v from C_i, let W_{ijk}^{abc} denotes the number of the pairs (x, y) such that $d(x, y) = a$, $d(v, y) = b$, $d(v, x) = c$, $x \in C_j$, and $y \in C_k$.
- The collection of all coefficients W_{ijk}^{abc} is known as the **interweight distribution** of the partition C with respect to the vertex v.



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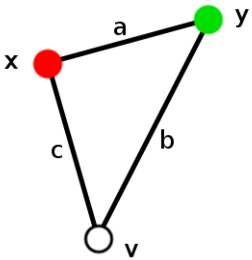

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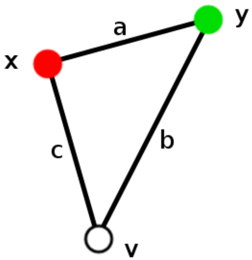

Strong distance invariance of equitable partitions

The **strong distance invariance** means that the interweight distribution $(W^{abc}_{ijk})_{a,b,c=0}^{d}{}_{j,k=1}^{m}$ of the partition with respect to a vertex v from C_i does not depend on the choice of v and depends only on the quotient matrix and the parameters of the distance regular graph.

Theorem (Vasil'eva, 2009)

The equitable partitions of the binary n-cubes are strongly distance invariant.

This statement does not hold for the distance regular graphs in general. Example: a union of three ternary 1-perfect codes.

$$S = \begin{pmatrix} 0 & 8 \\ 1 & 7 \end{pmatrix}$$

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How to calculate the interweight distribution?

- ! recursively (combinatorial arguments)
- ? direct formula (algebraic and combinatorial arguments ?)

Spectrum, multi-neighborhood

- For given partition $C = (C_1, \ldots, C_k)$, the **spectrum** of a vertex set X is the k-tuple $\mathrm{Sp}(X) = (x_1, \ldots, x_k)$, where $x_i = |X \cap C_i|$.
- In a natural way, the spectrum is generalized to multisets. If X is a multiset, then x_i is the sum over C_i of the multiplicities in X.
- The **multi-neighborhood** ΩX of a vertex set X is a multiset of the vertices of the graph, where the multiplicity of a vertex is calculated as the number of its neighbors from X.
- In other words, ΩX is the multiset union of the neighborhoods of the vertices from X.

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Spectrum of the multi-neighborhood

Lemma

Let C be an equitable partition (of an arbitrary graph) with quotient matrix S. For every vertex set X,

$$\mathrm{Sp}(\Omega X) = \mathrm{Sp}(X) \cdot S$$

Proof. From

$$A \cdot \overline{C} = \overline{C} \cdot S$$

we get

$$\overline{X} \cdot A \cdot \overline{C} = \overline{X} \cdot \overline{C} \cdot S$$

and

$$\overline{\Omega X} \cdot \overline{C} = \overline{X} \cdot \overline{C} \cdot S$$

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Calculating the usual weight distribution

- Let H^w be the set of weight-w vertices of the n-cube, $w = 0, 1, \ldots, n$. Let C be an equitable partition with quotient matrix S. Then

$$(\mathrm{Sp}(H^0), \mathrm{Sp}(H^1), \ldots, \mathrm{Sp}(H^n))$$

is called the weight distribution of C (with respect to $\overline{0}$).

- By Lemma, we have

$$\mathrm{Sp}(H^w) \cdot S = \mathrm{Sp}(\Omega H^w).$$

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Calculating the weight distribution

- $$\mathrm{Sp}(H^w) \cdot S = \mathrm{Sp}(\Omega H^w)$$

  The multi-neighborhood of H^w consists of H^{w-1} with multiplicity $n - w + 1$ and H^{w+1} with multiplicity $w + 1$.
- We conclude that

  $$\mathrm{Sp}(H^w) \cdot S = (n-w+1)\mathrm{Sp}(H^{w-1}) + (w+1)\mathrm{Sp}(H^{w+1}).$$
- Hence,

  $$\mathrm{Sp}(H^{w+1}) = \frac{\mathrm{Sp}(H^w) \cdot S - (n-w+1)\mathrm{Sp}(H^{w-1})}{(w+1)} = \mathrm{Sp}(H^0) \cdot \Pi^{w+1}(S)$$

  where Π^w is a polinomial of degree w.

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Calculating the weight distribution

$$\mathrm{Sp}(H^{w+1}) = \mathrm{Sp}(H^0) \cdot \Pi^{w+1}(S)$$

It is known that

$$\Pi^w(\cdot) = P_w(P_1^{-1}(\cdot))$$

where P_w is the Krawtchouk polynomial

$$P_w(x) = P_w(x; n) = \sum_{j=0}^{w} (-1)^j \binom{x}{j} \binom{n-x}{w-j}.$$

Calculating the weight distribution (algebraic way)

- $A \cdot \overline{C} = \overline{C} \cdot S$
- $A \cdot A \cdot \overline{C} = A \cdot \overline{C} \cdot S = \overline{C} \cdot S \cdot S$
- $A^i \cdot \overline{C} = \overline{C} \cdot S^i$ for every power i
- $\Pi(A) \cdot \overline{C} = \overline{C} \cdot \Pi(S)$ for every polinomial Π
- $\Pi^w(A) \cdot \overline{C} = \overline{C} \cdot \Pi^w(S)$ for $\Pi^w(\cdot) = P_w(P_1^{-1}(\cdot))$
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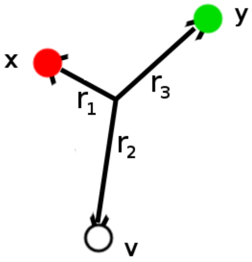
Integral interweight distribution

Let, for integer r_1, r_2, r_3, the (i,j,k)-entry of the 3-dimensional array $T^{r_1,r_2,r_3} = (T^{r_1,r_2,r_3}_{ijk})_{i,j,k}$ denotes the number of triples (v,x,y) of vertices such that

- $d(v,x) = r_1 + r_2$, $d(v,y) = r_1 + r_3$, $d(x,y) = r_2 + r_3$,
- $v \in C_i$, $x \in C_j$, $y \in C_k$.



Int. interweight distribution $\sim$ interweight distribution

If C as an equitable partition of the n-cube, then, because of the strong distance invariance,

$$W_{ijk}^{r_2+r_3,r_1+r_3,r_1+r_2} = \frac{T_{ijk}^{r_1,r_2,r_3}}{|C_i|}$$

Recursions for the integral interweight distribution

Theorem

Let $C = (C_1, \ldots, C_m)$ be an equitable partition of the n-cube with quotient matrix S. Then

$$\begin{aligned}
T^{r_1,r_2,r_3} \bullet_1 S &= (r_2{+}1)\,T^{r_1,r_2+1,r_3-1} + (r_3{+}1)\,T^{r_1,r_2-1,r_3+1} \\
&\quad + (n{-}r_1{-}r_2{-}r_3{+}1)\,T^{r_1-1,r_2,r_3} + \underline{(r_1{+}1)\,T^{r_1+1,r_2,r_3}},
\end{aligned}$$

$$\begin{aligned}
T^{r_1,r_2,r_3} \bullet_2 S &= (r_1{+}1)\,T^{r_1+1,r_2,r_3-1} + (r_3{+}1)\,T^{r_1-1,r_2,r_3+1} \\
&\quad + (n{-}r_1{-}r_2{-}r_3{+}1)\,T^{r_1,r_2-1,r_3} + \underline{(r_2{+}1)\,T^{r_1,r_2+1,r_3}},
\end{aligned}$$

$$\begin{aligned}
T^{r_1,r_2,r_3} \bullet_3 S &= (r_1{+}1)\,T^{r_1+1,r_2-1,r_3} + (r_2{+}1)\,T^{r_1-1,r_2+1,r_3} \\
&\quad + (n{-}r_1{-}r_2{-}r_3{+}1)\,T^{r_1,r_2,r_3-1} + \underline{(r_3{+}1)\,T^{r_1,r_2,r_3+1}},
\end{aligned}$$

where $T \bullet_1 S$, $T \bullet_2 S$, and $T \bullet_3 S$ denote the arrays with entries $\sum_{t=1}^{m} T_{tjk} S_{ti}$, $\sum_{t=1}^{m} T_{itk} S_{tj}$, and $\sum_{t=1}^{m} T_{ijt} S_{tk}$, respectively.

Proof

x	$[r_1{+}r_2,0,1]$	$\dots$	$\dots$	$[r_1{+}r_2,0,r_3]$	$\dots$	$\dots$	$v+\bar{1}$
$[r_1{+}r_2{-}1,1,0]$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$[r_1{+}1,r_2{-}1,r_3]$	$\dots$	$\dots$	$\dots$
$[r_1,r_2,0]$	$[r_1,r_2,1]$	$\dots$	$[r_1,r_2,r_3{-}1]$	$[r_1,r_2,r_3]$	$[r_1,r_2,r_3{+}1]$	$\dots$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$[r_1{-}1,r_2{+}1,r_3]$	$\dots$	$\dots$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$[1,r_1{+}r_2{-}1,0]$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	
v	$[0,r_1{+}r_2,1]$	$\dots$	$\dots$	$[0,r_1{+}r_2,r_3]$	$\dots$	$\dots$	$x+\bar{1}$

$$\begin{aligned}\Omega[r_1,r_2,r_3] = \quad & (r_1{+}1)\cdot[r_1{+}1,r_2{-}1,r_3] + (r_2{+}1)\cdot[r_1{-}1,r_2{+}1,r_3] \\ + \quad & (n{-}r_1{-}r_2{-}r_3{+}1)\cdot[r_1,r_2,r_3{-}1] + (r_3{+}1)\cdot[r_1,r_2,r_3{+}1]\end{aligned}$$

Recursions for the integral interweight distribution

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&\quad +(n-r_1-r_2-r_3+1)\,T^{r_1-1,r_2,r_3} + \underline{(r_1+1)\,T^{r_1+1,r_2,r_3}},
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\end{aligned}$$

Corollary

$$T^{r_1,r_2,r_3} = T^{0,0,0} \cdot \Pi^{r_1,r_2,r_3}(\bullet_1 S, \bullet_2 S, \bullet_3 S)$$

where Π^{r_1,r_2,r_3} is a polynomial of degree $r_1 + r_2 + r_3$ in three variables.

- Starting: $T^{0,0,0}_{iii} = |C_i|$, all other elements of $T^{0,0,0}$ are 0.
- **Problem.** To find an explicit formula for Π^{r_1,r_2,r_3}.

The polynomial

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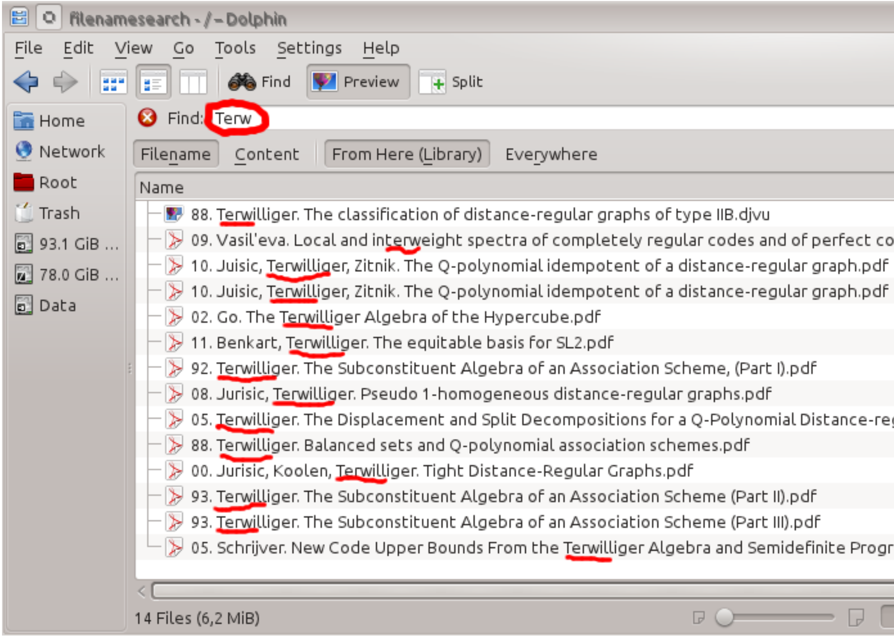
Connection with Terwilliger algebra



The correlation-immunity bound

Theorem (D. Fon-Der-Flaass, 2007)

If there exists an equitable partition of the n-cube with quotient matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $b \neq c$, then $c - a \leq n/3$ (the second eigenvalue of the matrix $\leq n/3$).

- **Empiric fact, $n \leq 36$.** For an array

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

  $n = a + b = c + d$, $b \neq c$, $c - a < n/3$, calculation of T^{r_1, r_2, r_3} gives negative values.
- **Problem.** Explain!

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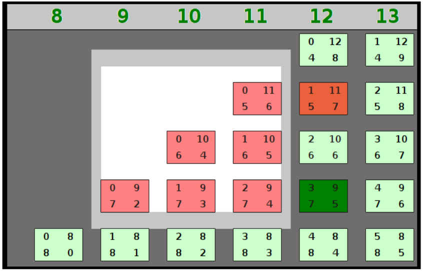
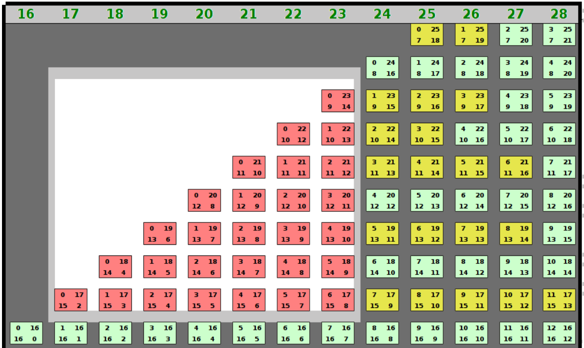
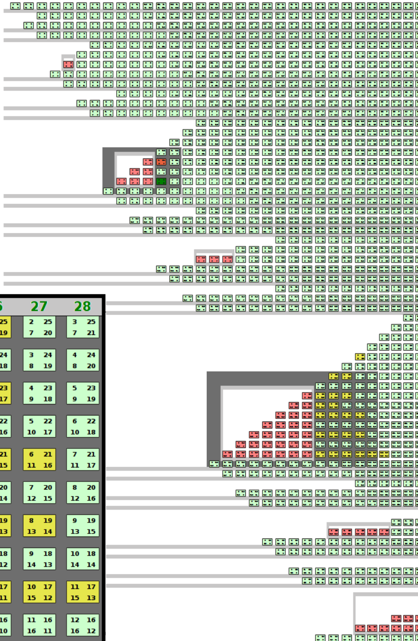
Quotient matrices







Thank you!

Thank you for your attention!

May The Force be with you!